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ON SOME ACADEMIC THEORIES OF MATHEMATICAL OBJECTS

In his critical study of Speusippus[1] Leonardo Tarán (T.) expounds an interpretation of a considerable part of the controversial books *M* and *N* of Aristotle's *Metaphysics*. In this essay I want to consider three aspects of the interpretation, the account of Plato's 'ideal numbers' (section I), the account of Speusippus' mathematical ontology (section II), and the account of the principles of that ontology (section III). T. builds his interpretation squarely on the work of Harold Cherniss (C.), to whom I will also refer. I concentrate on T. because he has brought the ideas in which I am interested together and given them a concise formulation; he is also meticulous in indicating the secondary sources with which he agrees or disagrees, so that anyone interested in pursuing particular points can do so easily by consulting his book.

The network of issues created by *M* and *N* is enormously complex. I have tried to isolate the specific issues with which I am concerned and to concentrate on the interpretation and evaluation of texts bearing specifically on those issues. I shall not question T.'s negative interpretation of the major texts used to turn Speusippus into a Neoplatonist, notably Iamb. *Comm. Math.* 15.6–18.12, Procl. *In Parmenidem*, ed. Klibansky-Labowsky, 38.32–40.7, and Arist. *Metaph.* 1092a11–17,[2] and I shall not make claims which turn on the acceptance or rejection of some of T.'s more questionable attributions of views to Speusippus, mainly those which depend on the principle that 'all the Aristotelian passages which mention the theory of the substantiality of the non-material point must be taken to be references to Speusippus' doctrine unless strong and unimpeachable evidence to the contrary can be adduced' (XXV).[3] In this way I hope to have minimized a difficulty which frequently arises with discussions in this area, namely the appearance that in order to discuss anything one has to discuss everything.

I. Ideal Numbers

T. begins his discussion of Plato's conception of number with an account of what he calls ideal numbers, the ideal two, the ideal three, etc.:

> These ideal numbers are not congeries of units, for each as an idea is a perfect unity which, like every other idea, has no parts, is not derived from any principles, and is not in any sense whatever the product of any other idea or element. . . . The ideal two and the ideal three, for example, are not respectively two units and three units, nor is the number five the sum of two and three. These numbers are just Twoness, Threeness, and Fiveness, each being a unity which is irreducibly itself and nothing else. (14)

This account of ideal numbers represents what might be called the received view (at least in the English-speaking world),[4] adumbrated by Cook Wilson in 1904[5] and adopted more or less *in toto* by Ross in his edition of the *Metaphysics*[6] and by C.[7] It is, of course, importantly inconsistent with Aristotle's characterization of Plato's view of number in *M* and *N*, but, according to T.,

[1] Leonardo Tarán, *Speusippus of Athens* (Leiden 1981). I refer to this work simply by page number.

[2] For a substantially different reading of these passages see Philip Merlan, *From Platonism to Neoplatonism*[3] (The Hague 1968) 98–140.

[3] For an example of a more flexible attitude on this question see Léon Robin, *La théorie platonicienne des idées et des nombres d'après Aristote* (Paris 1908) 229–232.

[4] The central work in what might be called the counter tradition is Julius Stenzel's *Zahl und Gestalt bei Platon und Aristoteles*[3] (Bad Homburg vor der Höhe 1959).

[5] 'On the Platonist doctrine of the ἀσύμβλητοι ἀριθμοί', *CR* xviii (1904) 247–260.

[6] See *Aristotle's Metaphysics* ii (Oxford 1958) 427.

[7] See *The riddle of the early Academy* (Berkeley and Los Angeles 1945) 34–37, or *Aristotle's criticism of Plato and the Academy*, i (Baltimore 1944) 513–517. (In the sequel I refer to these works as *Riddle* and *Criticism*, respectively.)

'there is direct evidence for it in the Platonic dialogues'. (14)[8] The evidence seems to me to justify only the following formulation of T.'s claim: there is a traditional interpretation of Platonic ideas as universals, and the received view of ideal numbers is one way of assigning to Plato an account of number compatible with that interpretation; moreover, nothing in the dialogues is any less consistent with the received view than certain passages are with the traditional interpretation.[9]

T. apparently takes for granted that the doctrine described by the received view is not only coherent, but, in some unspecified sense, true. However, there seems to me to be a certain obscurity even in the sentences already quoted. We might agree that Plato postulated what I will call numerical forms or ideas, twoness, threeness, etc., and that they have or should have the properties mentioned by T. But what justification is there for calling these numerical forms *numbers*? When T. writes, 'Plato's ideal numbers are . . . the necessary consequence of the theory of ideas; but *qua* numbers they are really the natural numbers' (14), what justification is there for the assumption that things which cannot be added together can be looked at *qua* numbers? T. and others have sought vindication for the received view in the logicist account of a number as a property or class of classes:

> Plato's ideal numbers are the hypostatization of the series of natural numbers. Unfortunately, this important conception of numbers was not understood by Speusippus, Xenocrates, and Aristotle, nor by the ancients generally. And so it was left to logicians and mathematicians in the nineteenth and twentieth centuries to rediscover, from points of view different from Plato's, the conceptual priority of the cardinal numbers. (15–16)[10]

The point remains that the logicist numbers can be added and subtracted, theorems can be

[8] The passages cited by T. (14), who also refers to Wilson, Ross, and C., are *Phd.* 96e–97b with 101b–c, *R.* 525c–526b, *Cra.* 432a–d, and *Phlb.* 56d–57a. For a clear statement of the relevant interpretation of the first three of these passages by C. see the first reference in the preceding note. There C. uses the first passage to show that Plato believed in a form corresponding to each number, a point which is not in dispute; the dispute is over the character of these forms. The second passage shows that λογιστική compels the soul to discuss numerical forms, but it also provides evidence that λογιστικοί study 'congeries of units', since Socrates imagines someone asking them, 'What kind of numbers are you talking about . . . in which each unit is all equal to every other, not differing in itself and having in itself no part at all?' C.'s attempt to construe 'each unit' to mean 'the unity of each of the numbers' strikes me as far-fetched, but, even if one accepts it, calling the passage evidence for the received view would, I think, be to confuse evidence for an interpretation with a reading of a passage based on the interpretation. The *Cratylus* passage, too, does not seem to provide any real evidence for the received view. C. uses it to describe how numbered groups might be said to fall short of numerical forms, namely by lacking their unity; but the *Philebus* passage mentioned by T. suggests another way: the units in numbered groups lack the absolute equality of the units in ideal numbers. In this respect the *Philebus* passage counts against the received view, but I am not confident that T. would really want to make use of it in this connection. C. ('Some war-time publications concerning Plato', *AJP* lxviii [1968] 189–191 n. 79) uses it as an argument against Plato's believing in 'intermediates', numbers sharing some properties with forms and some with numbered groups.

[9] Defenders of the received view usually treat Platonic ideas as concepts or universals (T.: 'For Plato the ideas are the hypostatization of all the universals.' [13] *Cf.* Sir David Ross, *Plato's Theory of Ideas* [Oxford 1951] 225), and the objections raised against the theory at the beginning of the *Parmenides* as to one degree or another insignificant. (See, e.g., Ross 87 or C., 'Parmenides and the *Parmenides* of Plato', *AJP* liii [1932] 135–138.) Others, of course, interpret the ideas as paradigmatic instances and find serious difficulties in the objections of the *Parmenides.* (See, for example, the papers by Gregory Vlastos and Peter Geach in R. E. Allen [ed.], *Studies in Plato's Metaphysics* [London and New York 1965]) Paradigmatic instances of numbers as conceived by the Greeks would almost certainly be 'congeries of units', but numerical universals presumably would not be. The scope of this paper precludes further discussion of the character of ideas in general, and I will content myself with trying to show that even if one accepts the notion that ideas are hypostasized universals, the received view of ideal numbers is ill-conceived.

[10] For expositions of the view T. evidently has in mind see, e.g., Bertrand Russell, *Introduction to mathematical philosophy* (London 1919) 1–19, or G. Frege, *The foundations of arithmetic*^{2}, translated by J. L. Austin (Oxford 1959). The applicability of the cardinal-ordinal distinction to Greek notions of number is doubtful (See Jacob Klein, *Greek mathematical thought and the origin of algebra*, translated by Eva Brann [Cambridge, Mass. and London 1968], vii, 46–60), but if one is going to apply it, the Greek conception of number as a 'congeries of units' is closer to a cardinal than an ordinal notion. (*Cf.* Ian Mueller, *Philosophy of mathematics and deductive structure in Euclid's Elements* [Cambridge, Mass. and London 1981] 69.)

proved about them. Perhaps this difference is what T. has in mind when he says that Plato's 'conception is essentially different from that of Frege and others' (16), but he still owes us an account of why, despite essential differences, Plato's numerical forms are numbers at all.

I do not intend to deny that the received view may offer a correct account of Plato's position; Plato may well have thought that numerical universals were numbers. I wish only to deny the existence of evidence that his doing so would represent an 'important conception of number' as opposed to a confusion, more refined than, but essentially of a piece with, the confusion of individual and universal with which Aristotle charges Plato. Perhaps the alleged failure of anyone in antiquity to understand this conception casts doubt on its viability, if not its reality. Perhaps too Aristotle's account of Platonic arithmetic numbers, intermediate between forms and sensibles and composed of homogeneous units capable of arbitrary combination ('in all probability the result of Aristotle's interpretation of Plato's theory' [15]), suggests that Plato himself recognized that he could not make plausible an identification of numerical forms and 'natural numbers'.

Thus far I have discussed what T. calls the cardinal aspect of ideal numbers. But they are also alleged to have an ordinal aspect:

> Plato also thought that the ideal numbers are an ordered series. This 'ordinal' aspect of the ideal numbers is not explicitly discussed in his works; but apart from incidental references to it in the dialogues, we may also infer it from the application to numbers of some general tenets of the theory of ideas. . . . Phenomenal numbers [i.e., collections of sensible things *qua* counted] . . . stand to one another in the relation of prior to posterior; we see this when we count. Since phenomenal numbers participate in, or imitate, the ideal numbers, it follows that the ideal numbers have a relation of prior to posterior to each other, a relation which must be independent of the fact that phenomenal numbers are congeries of units. Otherwise, according to Plato, we should not have been able to count. Aristotle refers more than once to the 'ordinal' aspect of the ideal numbers, explicitly stating that they stand in the relation of prior to posterior (Cf. *Metaph.* 1081b30–31 with 23–30, 33–35, 1080b11–12), that the ideal two is the first number, the ideal three the second number (Cf. *Metaph.* 1081b30–31, 1082b19–23), and that the Platonists did not posit a general idea of number, precisely because numbers stand to one another as prior to posterior (Cf. *EN* 1096a17–19). (14–15)[11]

The *Ethics* text cited, which is the only evidence that Plato did not posit a general idea of number,[12] refers only to numbers and not specifically to ideal or idea numbers. Presumably anyone who knows what numbers are knows that they are related as prior to posterior; Plato need not have taken this ordering as a special feature of ideal numbers. Moreover, if these numbers are simply numerical forms, there is no way to order them directly; twoness neither precedes nor follows threeness (although it would be easy enough to construct an order on the basis of an independent concept of number).

The *Metaphysics* passages all involve, directly or indirectly, Aristotle's own account of Plato's ideal numbers as collections of units, the units in each number being associable with one

[11] I have inserted the references to Aristotle from T.'s footnotes. I should perhaps mention that T. does not indicate the incidental references in the dialogues to which he refers.

[12] The passage says that Plato did not make there be an idea of numbers because he did not posit ideas in cases where one speaks of before and after. The passage has played a prominent role in the received view because, I suspect, Cook Wilson's reading of it was taken to solve a long-standing interpretive problem. (See C., *Criticism* 513.) For the passage had originally been read as asserting that Plato did not believe in numerical forms. The reading of the received view does eliminate a difficulty, but I know of no satisfactory reconciliation of the reading with the traditional interpretation of the theory of forms as universals, according to which 'to each kind of thing to which we apply a common name there corresponds a single idea' (13). C. (521) appears to argue that for Plato an idea of number in general would be redundant because for him 'each idea of number is . . . just its unique position as a term in the ordered series of numbers', so that an idea of number in general would be identical with this series. I find it quite unlikely that Plato would have thought of the idea of two as *just* the first position in the number series and not as, e.g., the property which all pairs share. But if he did, why should he have thought of the property of being a member of the series as identical with the series? Wouldn't the same considerations lead naturally to the view that a genus is identical with the collection of its species?

another but not with those in any other number. (Hereafter I call these numbers idea numbers.) T. wishes to reject this account, but to transfer to numerical forms the ordering relation which Aristotle ascribes to idea numbers. Such a manipulation of evidence strikes me as arbitrary. Moreover, there is reason to think that the references to priority and posteriority in the *Metaphysics* represent an Aristotelian formulation rather than an idea stressed by Plato. At 1080b11–14 Aristotle ascribes to Plato a belief in numbers with a before and after, the idea numbers, and intermediate arithmetic numbers. I think that T. and others have taken this casual reference to idea numbers as the ones with a before and after as an indication that priority-posteriority relations were closely and importantly linked with ideal numbers by Plato. However, Aristotle seems clearly to be thinking of the presence or absence of such relations as a way of contrasting idea numbers and intermediate ones. At first sight the contrast is curious since one would expect arithmetic number to be ordered just as idea number is. I take the point to be that, since there is one ideal two, three, and four, the ideal three has a unique predecessor and successor whereas an arithmetic three does not since there are infinitely many arithmetic twos and fours.[13]

The other passages cited by T. as Aristotle's explicit references to priority and posteriority in Plato's ideal numbers come at the beginning of *M.*6 where Aristotle offers a threefold schematization of possible views of separately existing numbers in which he makes central the notion that there is a unique instance of each number (there is a first in it and a subsequent one and each is different in species). According to the schematization either no units are associable (a view no one holds) or all are (and uniqueness vanishes, giving arithmetic number) or some are and some aren't, as in the doctrine of idea numbers. In connection with the last alternative Aristotle contrasts the counting of arithmetic number, which might be represented by *a*, *aa*, *aaa*, . . ., with the counting of idea numbers: *a*, *bb*, *ccc*, . . .

Aristotle, then, uses the idea of priority and posteriority to contrast one kind of congeries of units, idea numbers, with another, intermediate number. T. will assign neither kind of number to Plato and yet emphasize the significance of priority-posteriority among numerical forms where, as I have argued, it is inappropriate. Moreover, T.'s explanation of why Plato would want to insist on the ordering of numerical forms seems to me inadequate.[14] Why, for example, would Plato choose to invoke numerical forms to explain our ability to count but not our ability to add? It also seems clear that priority-posteriority alone will not explain our ability to count. I do not know that I should count two, three, four if all I know is that two is prior to three which is prior to four; I must also know that two immediately precedes three, and three four. But this knowledge seems inseparable from knowledge that a number is gotten from its immediate predecessor by adding one. And it is hard to think of someone with this knowledge of number not thinking of number as addible.

I conclude that if one is going to assign to Plato the theory of forms as hypostatized universals, one can assign him the doctrine of ideal numbers only if one is willing to admit that Plato was, in a significant way, confused. But there is no necessity to make this additional assignment. We can stick to numerical forms and take the *Ethics* passage as evidence that Plato postulated no idea of number because he recognized that 'phenomenal number' is related by priority and posteriority in the quite ordinary sense. However, it seems to me unreasonable to

[13] C. (*Criticism* 514) suggests that the absence of an order for arithmetic number depends on the fact that arithmetic numbers are [sometimes] related by inclusion. But there obviously are well-ordered series related in this way, the best known example being the von Neumann ordinals: φ, {φ}. {φ,{φ}}, . . . (Many textbooks in set theory include an account of these ordinals, usually with a heavy dose of mathematical symbols. Paul Bernays gives a more discursive presentation in 'A system of axiomatic set theory', in Gert H. Müller (ed.), *Sets and classes* [Amsterdam, New York, and Oxford 1976)] 19–24.)

[14] C. does not really give any explanation at all. He writes:

> These ideas of number are, as universals, ἀσύμβλητοι and, as ἀσύμβλητοι, entirely outside one another in the sense that none is part of another; thus (?) they form a series of different terms which have a definite order. (*Criticism* 514)

I have queried the 'thus' because for C. what precedes the semicolon is true of all ideas, but they do not constitute a series with a definite order.

invoke Aristotle's remarks about the priority of idea numbers while severing them from the specific, professedly Platonic doctrines with which he associates them.

II. Speusippus' Mathematical Ontology

In this section I will deal with Speusippus' view that arithmetic numbers and geometric objects (but not forms) exist alongside sensibles. For the aspects of this topic with which I will be concerned there is no major difference between geometricals and arithmeticals, and most of what T. says on the topic which is of interest to me in this paper refers to numbers. I shall therefore concentrate on them and make only occasional references to geometric objects.

According to T., Speusippus and Aristotle diverge from Plato by thinking of numbers as collections of homogeneous units. However, one clear difference between the two is that Aristotle 'rejects Speusippus' notion that mathematical number has separate existence. Aristotle maintains that numbers exist immanently in the sensibles and can be actualized only in thought, for they are "separable" only by abstraction' (17). This is a fairly standard account of Aristotelian doctrine with which I do not wish to quarrel here.[15] Rather I wish to indicate my misgivings which begin with a note on page 23 in which T. asserts that 'Aristotle does *not* ascribe to Speusippus Plato's alleged intermediate mathematicals'. There is an obvious and trivial sense in which this assertion is correct: Plato's alleged intermediates are intermediate between forms and sensibles, and Aristotle is very clear that Speusippus denied the existence of forms. T. makes this point, but he also makes the stronger claim that Speusippus' mathematicals do not satisfy Aristotle's characterization of intermediates according to which there are many of each kind, e.g., many equilateral triangles and many threes:

> [Aristotle] reports that Speusippus postulated the separate existence of mathematical numbers and magnitudes . . . and that he substituted mathematicals for Plato's ideas . . . as the unchangeable and separately existing objects of knowledge . . . All this *shows* that Speusippus' numbers and magnitudes are unique individuals. For example, there is only one separately existing mathematical 'three' and not many 'three's', as is the case with the intermediate mathematicals Aristotle ascribes to Plato.

I have italicized the word 'shows' because, although I find T.'s premisses acceptable, I can see no way in which the Aristotelian reports to which T. refers show what he takes them to show. And in the case of magnitudes I am not even clear how we are to understand the position assigned to Speusippus. Is there, for example, just one straight line, or is there a two-inch straight line and a three-inch one? And how can any straight line be unique in kind if there is, for example, a square contained by four equal ones? In any case, there seems to me to be a direct argument to show that T.'s claim is incorrect:

> Aristotle says that there are many instances of each kind of Platonic mathematical; (987b14–18)
>
> Aristotle contrasts Plato and Speusippus by saying that the former believed in forms and mathematicals whereas the latter believed in mathematicals only; (1076a19–22, 1080b11–16, 1086a2–13)
>
> therefore, Aristotle assigns to Speusippus a belief in mathematicals of which there are many of each kind.

The premisses of this argument are apparently acceptable to T., and the conclusion I have drawn seems to follow from them ineluctably. I shall, therefore, assume the conclusion and also that

[15] I have discussed Aristotle's mathematical ontology in 'Aristotle on geometric objects', *AGPh* lii (1970) 156–171; Jonathan Lear offers an alternative account in 'Aristotle's philosophy of mathematics', *PhR* xci (1982) 161–192. Aristotle actually says very little that is specific about the ontological status of numbers. He standardly speaks in a general way about mathematical ontology, and illustrates his views by reference to geometry.

Aristotle's attribution of the belief in such mathematicals is correct. As far as I am able to determine, my disagreement with T. on this point does not have far-reaching implications. For he does not invoke the conception of mathematicals unique in kind elsewhere, and the bulk of what he says about Speusippus' mathematical ontology would seem to apply equally to non-unique ones.

The note which I have been discussing is attached to a passage in which T. argues that Speusippus' conception of numbers as individual entities existing outside of time and place 'is impossible to reconcile with the notion that each number is a congeries of homogeneous and undifferentiated monads':

> If, for example, two and three are nothing but two and three such monads, how could these numbers have separate existence and be two different entities? For it appears that each such number is nothing apart from the component units; but it is impossible to differentiate the two units in two from any of the units in three. If numbers were nothing but congeries of abstract and undifferentiated monads, Aristotle would be right in thinking that these numbers cannot have separate existence but can be actualized only in thought. (23–24)

There appear to be two main claims in this argument: first, that a congeries of entities cannot be distinguished from the entities, and, second, that abstract homogeneous entities cannot be distinguished from each other. Both of these claims correspond to positions which have been adopted in the past and undoubtedly will be adopted in the future.[16] But they are by no means inevitable, and for understanding Speusippus' position it may help to explain why they are not.

The modern analogue of the claim that a congeries cannot be distinguished from the entities composing it is presumably that a set or class cannot be distinguished from the concatenation of its elements. Any modern philosopher who is a so-called mathematical realist or 'platonist' wishing to espouse the literal truth of mathematics has to reject this claim just because the distinction between a set and its elements is required for mathematics.[17] Similarly Speusippus could argue that the distinction between $2+3$ and 5 requires there to be a distinction between a number and its component units. The second claim should, I think, be given a stronger formulation than T. gives it. For the claim seems to be that homogeneous abstract units cannot exist separately because there is no conceivable difference betweeen them; in other words, it makes no sense to speak of distinct indistinguishable units; in other words, indiscernibles are identical.[18] It is important to see that this view should commit one to the position that abstract and undifferentiated units cannot exist in thought either. And something like this may have been Aristotle's position. For, on one reading, his doctrine of abstraction only entails that one can ignore differences between units, not that one can genuinely conceive undifferentiated units. The realist reply to this position is that it turns mathematics into a set of falsehoods, propositions which are true of things only when certain of their features are ignored; if the arithmetician speaks of distinct units with no differentiating properties, then, if arithmetic is true, there must be such units.[19]

I do not, of course, mean to imply that the realist position I am ascribing to Speusippus is

[16] A version of the first position is adopted by Nelson Goodman; see his *Problems and projects* (Indianapolis and New York 1972) 149–200. The second is a form of the principle that indiscernibles are identical associated first and foremost with Leibniz; for an exposition of Leibniz's view see, e.g., C. D. Broad, *Leibniz*, edited by C. Lewey (Cambridge 1975), 39–43.

[17] For arguments to this effect see, e.g., Hilary Putnam, *Philosophy of Logic* (New York, Evanston, San Francisco, and London 1971). The issues involved here are clarified when one talks about congeries of congeries, a level of abstraction which Greek mathematics does not seem to have reached. Modern mathematics is unthinkable without the distinction between a set of numbers and a set of sets of numbers. For traces of the raising of the analogous issue for numbers in antiquity see Aristotle's suggestion at *Metaph.* 1044a2–5 that a number should not just be a σωρός, and Socrates' argument at *Tht.* 204b–205a that there is no difference between τὸ ὅλον and τὸ πᾶν.

[18] Modern discussions of the indiscernibility of identicals have focused on physical rather than mathematical objects. The notion of indiscernible non-identicals had a substantive role to play in philosophy of mathematics only as long as integers were thought of as sets of units.

[19] For a Neoplatonic example of this kind of reasoning see Syrian. *in Metaph.* 90.9–15.

unassailable, but only that it is tenable. Aristotle's doctrine of abstraction and its revisionist account of mathematical truth have their attractions. And the ἀπορία which he brings to bear against intermediates, namely that there is no justification for being a realist in connection with arithmetic and geometry but not in connection with astronomy, harmonics, and optics, has a certain force in the context of Greek science.[20] However, these arguments are not adequate to force abandonment of the position that the paradigms of Greek science, arithmetic and geometry, must be interpreted as applying to exactly the kind of objects they invoke.[21]

III. Speusippus' Mathematical Principles

The center of T.'s interpretation of Speusippus' conception of principles is our one substantial fragment of Speusippus' own work, a passage of 58 lines (83.6–85.9) from the anonymous *Theologoumena Arithmeticae*, a work of quite uncertain date. The passage is said to be a quotation from Speusippus' *On Pythagorean numbers*, and consists of a series of artificial and superficial reasons for praising the number ten. T. reads out of this passage a (for the Greeks) highly original conception of one as a number and a very pedestrian conception of a principle. He then uses these conceptions to rule out things Aristotle says explicitly or implicitly about Speusippus which are incompatible with them. I will raise some questions about T.'s reading of the fragment of *On Pythagorean numbers*, but first I want to raise a more general methodological issue.

The principle of giving an author's words primacy over reports about what he said is well established in cases where there is good reason to suppose that the words and the reports concern the same subject. It should not be necessary to argue that the principle is not well established in other cases, notably in the case of Plato's words and Aristotle's reports. However, we presumably have all of Plato's published work. In the case of Speusippus we are dealing with sixty-odd lines from a considerable output.[22] Moreover, the content of the lines is trivial although Speusippus was probably 'a philosopher of considerable originality'. (108) *Prima facie* it seems unlikely that many scholars will be inclined to accept an interpretation built on these few lines at the expense of much of what our major source for Speusippus' views, his contemporary Aristotle, says.[23] This general point seems to me to be of considerable importance, although, to be sure, evaluation of any interpretation must be based on the reading of particular texts.

[20] *Metaph.* 997b12–24. For the reasons underlying my claim see my paper 'Ascending to problems' in John Anton (ed.), *Science and the sciences in Plato* (Albany 1980) 103–121.

[21] In connection with his rejection of Speusippus' conception of number as untenable T. cites one Aristotelian argument:

> It is strange that there should be some one which is a first of ones as they [= Speusippus] say, and not a two which is a first of twos or a three of threes. For all are subject to the same argument. If this is the way things are in the case of number and one postulates that only mathematical number exists, the one is not [i.e., should not be] a principle (for such a one must differ from the other units, and, if so, there must be some two which is a first of twos and similarly for the other succeeding numbers). But if the one is a principle, the facts about number must be as Plato used to say, and there must be a first two and three, and the numbers must not be associable. (1083a24–34)

T. analyzes this argument as follows:

> [Aristotle] implies that the very notion of separately existing mathematical numbers is a contradiction in terms. If numbers have separate existence, there must be not only a first One, as Speusippus said, but also a first Two, a first Three, etc. *For to exist apart any number would have to be a different entity from any other separately existing number, and this would mean that each number is 'incomparable' with every other number.* Mathematical number, however, cannot be incomparable, since the component units are all comparable and undifferentiated. (24)

I find nothing in Aristotle's text corresponding to the sentence I have italicized. Nor do I see how the first half of this sentence could 'mean' what T. says it does, since the first half only says that separately existing things are different from one another. In fact Aristotle appears to be arguing simply that if Speusippus is going to postulate a first one as a principle of number, he ought, by parity of reasoning, to assert a first two, etc. But such a first two is incompatible with belief in mathematical numbers only since among them there is no first two; belief in a first two gets one back into Plato's idea numbers. What Aristotle says does not rule out the possibility of believing in separately existing mathematical and idea numbers.

[22] Some thirty titles in incomplete list of Diogenes Laertius (iv. 4–5).

[23] T. does offer explanations of Aristotle's alleged mistakes and misunderstandings, but these rarely

Aristotle's descriptions of the outlines of Speusippus' views are reasonably clear. Speusippus denied the existence of forms and made numbers the first kind of entities, geometrical objects the second. He derived numbers from a formal principle, the one, and a material principle, multiplicity (πλῆθος); for geometric objects he made the point the formal principle and something like multiplicity but distinct from it the material principle. We do not know the name of Speusippus' 'material' principle for geometricals, but *Metaph.* 1085b27–34 suggests that it may have been διάστημα (interval, extension, dimension, distance). In any case I shall use this term to simplify exposition.[24]

In the fragment of *On Pythagorean numbers* Speusippus refers to principles three times:

1. 1 is a point, 2 a line, 3 a triangle, 4 a pyramid, and all these are primary and principles of each of the things homogeneous with them.
2. These things are primary among (ἐν) planes and solids: point, line, triangle, pyramid.
3. The first principle of magnitude is point, the second line, third surface, fourth solid.

T. describes these passages as showing that for Speusippus the point is a magnitude, albeit a paradoxical one without dimension. (37) He later says of 1 and 2, 'This means that the point is the first minimal magnitude (though it is a "magnitude" without dimension), the line the second, the triangle, which is the first plane, the third, and the pyramid, which is the first solid, the fourth'. (45) T. explains the primacy of triangles by reference to the fact that 'any plane can be divided into triangles while the triangle cannot be divided into anything but triangles', (45) and suggests an analogous account for pyramids. 'Plane' in this explanation should mean 'plane rectilineal figure', and it seems reasonable enough to suppose that plane rectilineal figures are what Speusippus has in mind as the things homogeneous with triangles and pyramids. This supposition rather restricts Speusippus' conception of magnitudes, but there is, as far as I know, no evidence for his working with any broader conception.

Speusippus' association of 2 with the line suggests that he is thinking only of the straight line, which is determined by two points as the triangle and pyramid are determined by three and four points, respectively. However, the only things homogeneous with straight lines in the way rectilineal plane figures are homogeneous with triangles are straight lines, and, of course, the same thing can be said, in a somewhat weaker sense, of points. Thus the 'doctrine' of principles implicit in 1 commits Speusippus to the view that the point is principle of the point, and the straight line is principle of the straight line. Rather than assign such a triviality to Speusippus it seems to me preferable to acknowledge that Speusippus' real interest in 1 is to glorify the number ten by invoking the presumably well-known correlation of elementary geometric objects with the first four positive integers, which sum to ten.[25] To call these objects principles is to stress their significance and elementary nature, but not necessarily to have in mind a specific account of principles.

The same, it seems to me, can be said of 2. For Speusippus goes on to do some arbitrary counting of points and lines in various triangles and pyramids to get some more tens. 2, unlike 1, could be taken to imply that points are magnitudes, but only if one is willing to ascribe to Speusippus the extremely odd view that a point is plane or solid, an idea hardly compatible with the notion of homogeneity invoked in 1.

strengthen one's confidence that an error has been found. I do not intend to glorify Aristotle's skills as an expositor of the views of others. But even after finding him guilty on many counts of misrepresentation, I am reluctant to assume him generally guilty until proven innocent. However, without this assumption many particular charges of guilt seem hollow. On the other hand, the assumption has the value of consistency. Without it putting together an interpretation of Aristotle's accounts of the views of others with as few loose ends as, say, that of C., is very difficult, perhaps impossible. But to minimize the loose ends of an interpretation and maximize its consistency is not always to maximize its plausibility.

[24] Nothing turns on the choice of this term. *Cf.* T.'s remark on *Metaph.* 1085b27–34 (362). He, however, never gives more than conditional assent to Aristotle's ascription of 'material' principles to Speusippus.

[25] The various elements of the correlation crop up rather frequently in Aristotle; see, e.g. *Metaph.* 1084b26–27, 1090b21–23.

T.'s understanding of 3, which likewise implies nothing about points being magnitudes, is not clear to me. After calling it 'important', he says that 'two inferences can be drawn'. (45) However, both inferences appear to be based on 1 and 2: first, that to be a principle of something means to be the first minimal entity of its kind, and, second, that there are several 'formal' principles of magnitude, namely point, line, triangle, and pyramid. 3 is introduced by the words 'And the same thing holds for γένεσις', but unfortunately it marks the end of the Speusippus fragment. Γένεσις suggests something more like the ontological principles discussed by Aristotle in *M* and *N*, and point, line, surface, and solid sound much more like point and διάστημα than do point, line, triangle, and solid. I have argued elsewhere[26] for a possible correlation between Aristotle's account of Speusippus' geometric principles and 3, the idea being that the point serves as a determinant of form (and hence as a formal principle in one Aristotelian sense) of (rectilineal) figures in one, two, and three dimensions, dimensionality serving as a material principle in Aristotle's sense. This interpretation is not built on the fragment alone and it does discard significant statements by Aristotle as misinterpretations or verbal quibbling, but it seems to me to produce a plausible and reasonable alternative account to the one offered by T. which I have described here.

Whatever the correct interpretation of Speusippean geometric principles is, the fragment from *On Pythagorean numbers* does not justify the claim that for Speusippus the point was an odd kind of magnitude. 1 may show that Speusippus used the word 'principle' to refer to minimal or simplest entities of a given kind, but I am reluctant to think that he would have called points minimal points or straight lines minimal straight lines. In any case 3 shows that Speusippus did not necessarily use the word 'principle' to refer to minimal entities of a given kind.

For T. the idea that Speusippean principles are minimal entities dovetails with his claim that for Speusippus one is a number and hence a principle of number in the appropriate sense. This claim is based on Speusippus' strictly numerical arguments for the greatness of the number ten:

> Ten contains equally many odds [1, 3, 5, 7, 9] as evens [2, 4, 6, 8, 10]; it has equally many primes [1, 2, 3, 5, 7] as composites [4, 6, 8, 9, 10]; and it has as many submultiples as multiples.

The last of these considerations is problematic. The ordinary multiples in ten are 4, 6, 8, 9, 10, and the submultiples are 2, 3, 4, 5. Speusippus drops 4 because it appears on both lists, and 7 because it appears on neither. To make up the remaining imbalance he must add 1 to the submultiples and ignore the fact that doing so makes every number a multiple. This difficulty suggests that Speusippus does not have a well worked out doctrine that one is a number, but that he is simply engaged in arbitrary manipulation to make things come out right.[27] Analogous discussions in later writers involve vacillation between denying that one is a number and ascribing to it properties usually assigned to numbers only. For example, Theon denies that one is a number when its being a number would make all pairs of numbers have a common factor (*Expositio* 24.23), but then (25.22–24) treats it as odd when he wants to represent two as an even-times odd number. Even Euclid, who normally observes the distinction between the unit and numbers, applies *Elements* vii 12, a theorem established for numbers only, to units in the proof of vii 15. Speusippus may, of course, have been generally more consistent than others in his treatment of one, but a number of factors make me quite doubtful: his wobbling in the very passage where he does treat one as odd, prime, and a submultiple; the very arbitrary character of that passage; the absence of any other confirmatory evidence; and, finally, the fact that Aristotle does not mention Speusippus' unusual treatment of the one and treats the Speusippean one as a formal principle.

[26] In a paper called 'Aristotle's approach to the problem of principles in *Metaphysics M* and *N*' to be published in the proceedings of the 1984 Symposium Aristotelicum.

[27] It should perhaps be pointed out that Speusippus never explicitly calls one a number, but merely says things which imply it to be odd, prime, and a submultiple. T. takes for granted that ascribing these properties to one necessitates thinking of it as a number, a plausible but not absolutely compelling assumption.

My own view of the Speusippean one is in many ways quite close to T.'s. I believe that the one is simply the unit or, more precisely, units, and that multiplicity is like a set-forming operation which generates numbers from the units. T. himself thinks it 'likely that Speusippus . . . used "the One", "one", and "monad" indistinctly', but he also believes that the only genuine alternative to his account is to treat Speusippus' one as 'merely conceptual unity' without separate existence. (35) This belief is presumably connected with his rejection of the realistic conception of mathematical units as untenable, a rejection which I have already argued is unwarranted. However, my purpose in this concluding section has not been to argue for an alternative to T.'s account of Speusippean principles, but simply to indicate that one is available while showing why I think T.'s use of the fragment of *On Pythagorean numbers* is unjustified.

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